Synchronization of chaos using continuous control

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We show that two identical chaotic systems can be synchronized by applying a method of continuous chaos control. The presented method is especially useful for higher-dimensional systems.

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The essential property of a chaotic trajectory is that it is not asymptotically stable. Closely correlated initial conditions have trajectories which quickly become uncorrelated. Despite this obvious disadvantage, it has been established that synchronization of two chaotic systems is possible [1–3] and has potential practical applications in secure communications [4,5].

Pecora and Carroll showed [1] that there exists the class of chaotic systems for which synchronization can be achieved. Consider a system that can be divided into drive subsystem (whose largest Lyapunov exponent is positive) and driven subsystem (with all negative Lyapunov exponents). In this case trajectories from two identical driven subsystems can be synchronized if the same driven system is used. This result has been numerically and experimentally verified mainly on electrical systems [1–3].

de Sousa Vieira, Lichtenberg, and Lieberman [2] showed that the boundary of possible synchronization and nonsynchronization is strictly connected with the transition from chaotic to hyperchaotic behavior that is characterized by at least two positive Lyapunov exponents [6,7].

Recently chaos controlling method of Ott, Grebogi, and Yorke (OGY) [8] has been used to synchronize chaotic systems [9]. It was shown that by applying small, judiciously chosen, temporal-parameter perturbation to one of the chaotic system we can stabilize its orbit around a chaotic trajectory of the other system achieving synchronization of the two systems. The OGY method has been applied to control chaotic orbits by Mehta and Henderson [10]. Although this idea can be directly applied to synchronization problems it may not be easily used for dynamical systems more general than the one considered in Ref. 10.

The OGY method requires a permanent computer monitoring of the state of the system and deals with Poincare map to evaluate the changes of the parameter. Since the corrections of the control parameter are rare and small, the fluctuation noise leads to occasional bursts of system into the region far from the desired orbit causing breaks in synchronization. The frequency and duration of these breaks increase with the increase of noise intensity.

In this Brief Report we discuss the possibility of applying a continuous chaos controlling method developed by Pyragas [11] to achieve a synchronization of two chaotic systems. The method we developed is a generalization of the method of controlling dependence on initial conditions [12].

To synchronize two chaotic systems,

$$\dot{x}=f(x)\ , \tag{1a}$$

$$\dot{y}=f(y)\ , \tag{1b}$$

where $x, y \in \mathbb{R}^n$, that we call A and B, we use the strategy which is schematically illustrated in Fig. 1. We assumed that some state variables of both systems A and B can be measured. Let us say that we can measure signal $x_i(t)$ from the system A and signal $y_i(t)$ from B $(i=1,2,\ldots,n)$. Chaotic systems A and B are coupled unidirectionally in such a way that the difference $D(t)$ between the signals $x_i(t)$ and $y_i(t)$ is used as a control signal

$$F(t)=K[x_i(t)-y_i(t)]=KD(t) \tag{2}$$

introduced into one of the chaotic systems (A in Fig. 1) as a negative feedback. $K>0$ is an experimentally adjustable weight of the perturbation and we discuss its selection later. An experimental realization of such a feedback presents no difficulties for many practical systems. The perturbation signal (2) modifies the solution of Eq. (1a) and forces synchronization. When the synchronization is achieved, i.e., $x_i(t)=y_i(t)$, $F(t)$ becomes zero and the chaotic systems A and B become practically uncoupled. This property shows that in the synchronized regime perturbation (2) does not change dynamics of chaotic systems A and B.

We illustrate our synchronization procedure on the example of two identical Duffing's equations

$$\ddot{x}+a\dot{x}+x^3=B\cos t\ , \tag{3a}$$

$$\ddot{y}+a\dot{y}+y^3=B\cos t\ . \tag{3b}$$

It is well known that for $a=0.1$, $B=10.0$ Eqs. (3) show chaotic behavior [13] and if $\mathbf{x}(0)$ is slightly different than $\mathbf{y}(0)$ ($\mathbf{x}=[x,\dot{x},1]^T$, $\mathbf{y}=[y,\dot{y},1]^T$) trajectories $\mathbf{x}(t)$ and $\mathbf{y}(t)$ diverge exponentially from each other. To achieve synchronization we add perturbation signal (2) to Eq. (3a) obtaining a coupled system

$$\begin{gathered}\ddot{x}+a\dot{x}+x^3=K(y-x)+B\cos t\ ,\\ \ddot{y}+a\dot{y}+y^3=B\cos t\ .\end{gathered} \tag{4}$$

In numerical calculations we used the fourth-order

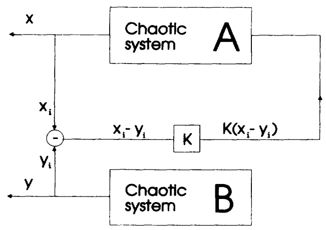

FIG. 1. Scheme of our method of synchronization of two chaotic systems. Some dynamical variables of two systems $[x_i(t)$ and $y_i(t)]$ are measured and chaotic systems A and B are coupled via negative feedback.

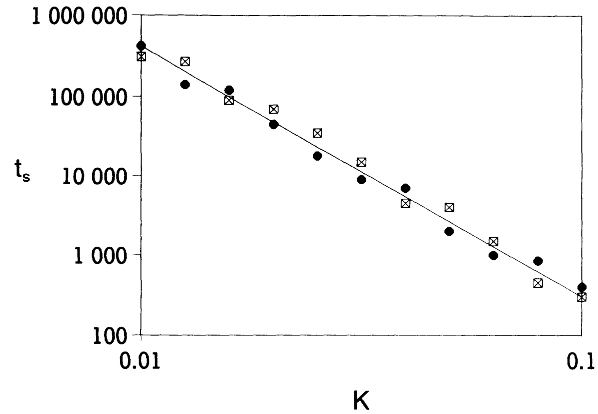

FIG. 2. Averaged time to achieve synchronization $\langle t_s \rangle$ versus coupling stiffness K; ● systems without noise, ⊠ white noise with amplitude 10^{-3} added to the chaotic system B.

Runge-Kutta method with integration step $\pi/200$. Lyapunov exponents have been estimated using Wolf *et al.* method [14]. We have studied the dependence of the synchronization time t_s, defined as the time taken to reach the synchronization with the assumed precision 10^{-4}, i.e., $|y(t)-x(t)|<10^{-4}$. For K in the interval (0.01, 0.1) we estimated t_s for 1000 randomly chosen initial conditions and averaged them. Our results are shown in Fig. 2 (dots) where as expected, t_s decreases when K increases. To verify the effectiveness of our method in noisy situation we add additive white noise to one of the chaotic systems—B. We found that synchronization is still possible and that the noise has no influence on the synchronization time t_s as shown in Fig. 2 (squares). The interesting and practically useful features of our method is that the continuous control allows automatic corrections of perturbation signal (2) [through $y_i(t)$] when noise is added to the system. No other modification are necessary for application of our method to noisy systems.

The presented example showed that our method is a very convenient way to synchronize multidimensional systems by feeding back a single variable. Generally it has to be one of the state variables described by a drive subsystem (in the classification of Pecora and Carroll [1]) of chaotic systems A and B as feeding back variables from driven subsystem gives no results in continuous chaos control method [15]. Coupling stiffness K has to be chosen in such a way to achieve small synchronization time t_s. However, due to the limitations of continuous chaos control methods [15] feeding back one state variable is not always successful. One can easily show that synchronization can be achieved only if the number of positive Lyapunov exponents of the coupled system equals the number of positive Lyapunov exponents of a single system. In our example only one positive Lyapunov exponent in the spectrum of Eqs. (4) is allowed to achieve synchronization. Knowing equations of chaotic systems A and B we can easily check the above condition by direct computation of Lyapunov exponents. If the systems A and are given only by time series we can check properties of correlation dimension (one scaling region for chaotic attractor and at least two scaling regions for hyperchaos) [7]. If Lyapunov exponents condition is fulfilled the coupled system (4) evolves on the same manifold on which both chaotic systems evolve and that is why synchronization can be obtained. When it is not fulfilled the coupled system evolves on higher-dimensional manifold on which hyperchaotic attractor exists and according to de Sousa Vieira, Lichtenberg, and Lieberman [2] synchronization cannot be obtained. As shown in Ref. 14 by weakly coupling two chaotic systems, as in our synchronization procedure it is not so easy to find hyperchaos. But if we are unlucky to find more positive Lyapunov exponents for the coupled system than we have in the original systems we can try to avoid hyperchaos either by changing coupling stiffness K or by simultaneously feeding back more state variables of chaotic systems A and B.

To summarize, we presented a method for achieving synchronization of two chaotic systems by applying a continuous chaos control scheme. Our method does not require the monitoring of chaotic trajectories or application of targeting procedure that was necessary in the previous method [3]. Due to the continuity of the control, our synchronization method is also efficient in the presence of noise and can be easily applied to the experimental systems, especially to secure communication systems [4,5].

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